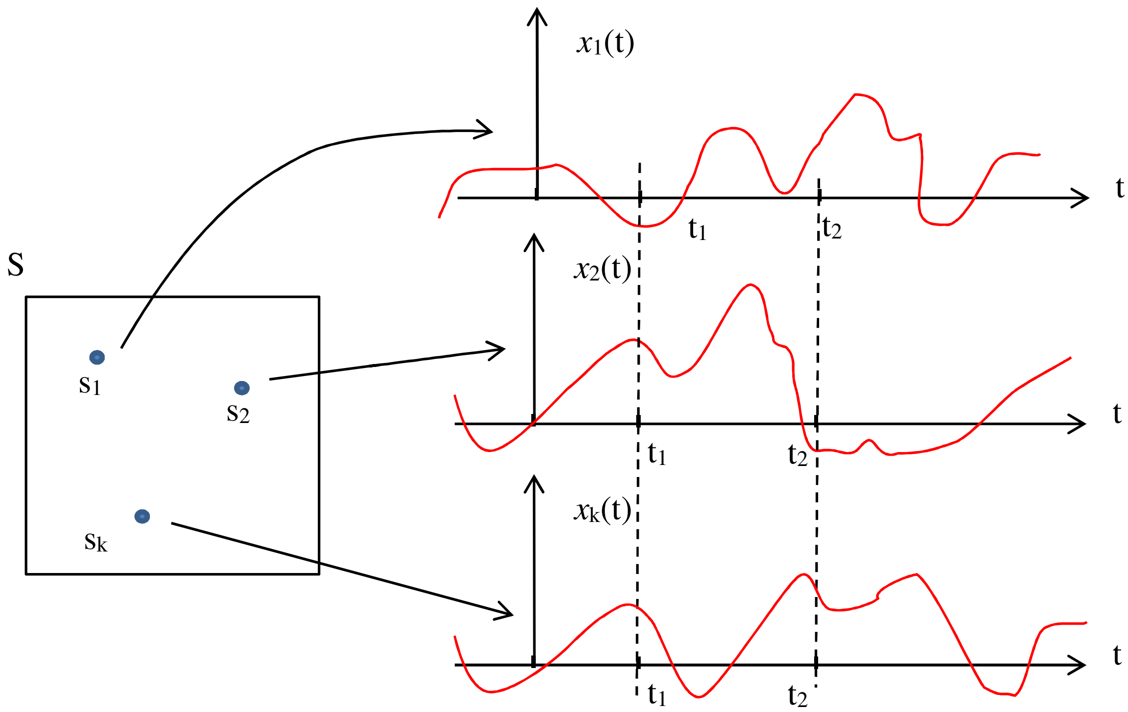
Chapter 10

Random Processes

1. Introduction

- In chapter 2, we have defined a random variable X as a mapping of the elements of the sample space S into points on the real axis.
- For random processes, the sample space could map into a family of time functions. That is if every outcome S is a function of time. This will lead to the concept of a random process
- **Definition of a random process: A** random process $X(t)$ is am mapping of the elements of the sample space into functions of time. Each element of the sample space is associated with a time function as shown below.



- Associating a time function to each element of the sample space results in a family of time functions called the **ensemble**.

Example 1

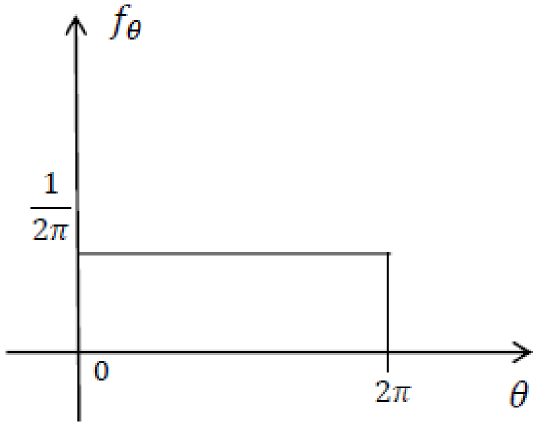
Consider a random process $X(t) = Acos(\omega t + \theta)$, where θ is a random variable uniformly distributed between 0 and 2π.

Show some sample functions of this random process.

What could we say on the type function above if for example we fix the phase $\theta = \pi/4$

Solution:

$$f_\theta(\theta) = \begin{cases} \frac{1}{2\pi} & 0 \le \theta \le 2\pi \\ 0 & \text{otherwise} \end{cases}$$



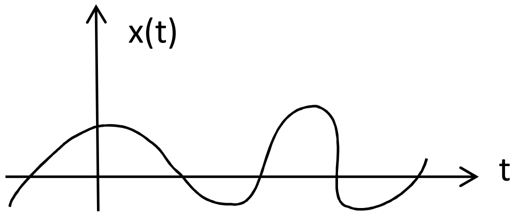
- Sample functions of this r.p. are cosine functions with frequency wt and different phase shifts when t is fixed.

- Fixing θ and varying t results in a deterministic time function.

2. Classification of random processes

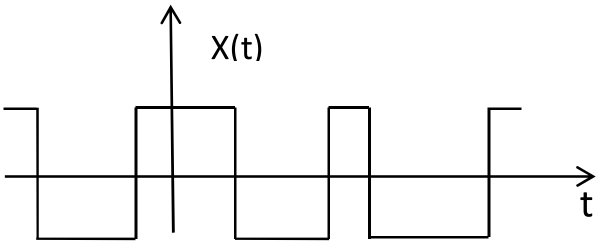
2.1 Continuous random process

- In this case, both X(t) and t have continuous values



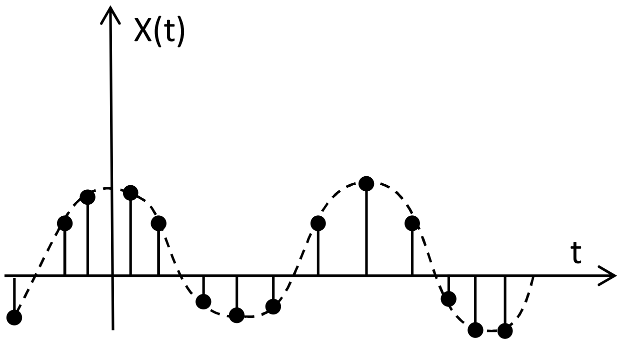
2.2 Discrete random process

- $X(t)$ assumes a discrete set of values while time t is continues



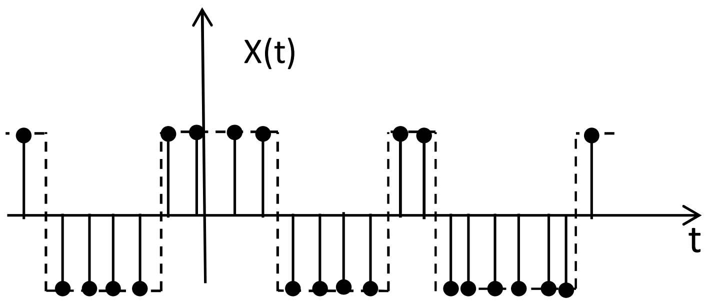
2.3 Continuous random sequence

- $X(t)$ assumes a continuous set of values while time t is discrete



2.4 Discrete random sequence

- Both $X(t)$ and t assumes a discrete set of values



- Fixing the time t, the random process $X(t)$ becomes a random variable. In this case, the techniques we use with random variables apply.

- We can characterize a random process by the first order distribution as:

$$F_X(x;t) = P[X(t_0) \leq x]$$

- The first order density function for all possible values of *t* is given by:

$$f_X(x;t) = \frac{d}{dx} F_X(x;t)$$

- The second order distribution function is the joint distribution of the two random variables $X(t_1)$ and $X(t_2)$ for each t_1 and t_2. It is given by:

$$F_{X_1 X_2}(x_1, x_2; t_1, t_2) = P[X(t_1) \leq x_1 \ \ and \ X(t_2) \leq x_2]$$

- The second order density function is given by:

$$f_{X_1 X_2}(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_{X_1 X_2}(x_1, x_2; t_1, t_2)$$

Deterministic and Nondeterministic Process:

- A r.p. (random process) is called **deterministic** if future values of any sample functions can be predicted from past values. For example:

$$X(t) = A\cos(w_0 t + \theta)$$

Here, A, θ, ω_0 (or all) may be r.v.s.

- A r.p. is called **nondeterministic** if future values of any sample function cannot be predicted exactly from past values.

3. Independence and stationarity

- In many problems of interest the first and second order statistics may be necessary to characterize a random process.
- The **mean value** (in general function of time) of a random process $X(t)$ is given by:

$$E[X(t)] = \int_{-\infty}^{+\infty} x f_X(x;t)dx$$

- The **autocorrelation function** is defined by:

$$R_{XX}(t_1,t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2\, f_{X_2X_1}(x_1,x_2;t_1,t_2)dx_1dx_2$$

3.1 Statistical Independence:

Two random processes X(t) and Y(t) are **statistically independent** if the random variable group X(t_1),…… X(t_N) is independent of the group Y($\grave{t_1}$),……… Y($\grave{t_M}$) for any choice of $t_1,\dots,t_N,\grave{t_1},\dots,\grave{t_M}$:

$$f_{X,Y}(x_1,\dots,x_N,y_1,\dots,y_M;t_1,\dots,t_N,\grave{t_1},\dots,\grave{t_M})$$

$$= f_X(x_1,\dots\dots..x_N,t_1\dots\dots..t_N).f_Y(y_1,\dots\dots..y_M,\grave{t_1}\dots\dots..\grave{t_M})$$

3.2 First-order stationary processes:

- A random process is called **first order stationary in the strict sense** if the first-order density function does not change with a shift in time :

$$f_X(x_1;t_1) = f_X(x_1;t_1+\Delta) \quad where \qquad \Delta \text{ is any real number}$$

- This means that $f_X(x_1, t_1)$ is independent of t_1 and the process mean value $E[X(t)]$ is constant:

$$E[X(t)] = \int_{-\infty}^{\infty} x\, f_X(x;t)dx = constant$$

3.3 Second-order stationary process

- A random process is **second order stationary in the strict sense** if the first and second order density functions stratify:

$$f_X(x_1; t_1) = f_X(x_1; t_1 + \Delta) \quad (1)$$

$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_1 + \Delta, t_2 + \Delta) \quad (2)$$

- This means that f_X is a function of time difference $t_2 - t_1$ and not absolute time.
- A second-order stationary process is also first-order stationary
- For second-order stationary random process , the autocorrelation function is a function of time differences and not absolute time :

$$\tau = t_2 - t_1$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)] = R_{XX}(\tau)$$

3.4 Wide sense stationary random process

- As the basic conditions for strict sense stationary process are usually difficult to verify (equations (1) and (2))
- We often resort to a weaker definition of stationarity known was wide sense stationary.

- Thus, when the autocorrelation function $R_{XX}(t_1, t_2)$ of the random process $X(t)$ varies only with the time difference $|t_1 - t_2|$ and the mean $E[X(t)]$ is constant then $X(t)$ is said to be wide sense stationary. That is:

$$(1)\quad E[X(t)] \equiv constant$$

$$(2)\quad R_{XX}(t+\tau, t) = R_{XX}(\tau)$$

- A strict sense stationary process is wide sense stationary process but the inverse is not true.

Example 2

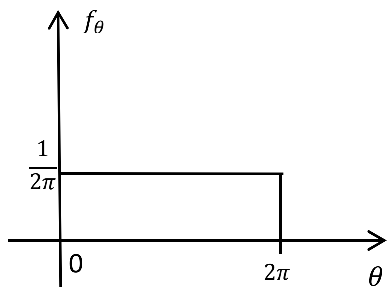
Consider a random process $X(t) = Acos(\omega t + \theta)$, where θ is a random variable uniformly distributed between 0 and 2π. Is the process stationary in the wide sense?

Solution:

$$E[X(t)] = \int_0^{2\pi} A\,cos(w_0 t + \theta).\frac{1}{2\pi}d\theta = \frac{A}{2\pi}\sin(w_0 t + \theta)\Big|_0^{2\pi} = 0$$

$$R_{XX}(t, t+\tau) = E[X(t)X(t+\tau)]$$

$= E[A\,cos(w_0 t + \theta)A\,cos(w_0 t + w_0\tau + \theta)]$. We have:



$$\cos u \cos v = \frac{1}{2}[\cos(u - v) + \cos(u + v)]$$

$$= \frac{A^2}{2}\,E[\cos(w_0\tau) + \cos(2w_0 t + w_0\tau + 2\theta)]$$

$$= \frac{A^2}{2}\cos(w_0\tau) + \frac{A^2}{2}\,\mathrm{E}[\cos(2w_0 t + w_0\tau + 2\theta)]$$

$$= \frac{A^2}{2}\cos(w_0\tau) = R_{XX}(\tau)$$

X(t) is wide-sense stationary r.p.

3.5 Jointly wide-sense stationary:

- For two random processes *X*(*t*) and *Y*(*t*), we say they are jointly wide-sense stationary if each process is wide-sense stationary and:

$$R_{XY}(t, t+\tau) = E[X(t)Y(t+\tau)] = R_{XY}(\tau) \Rightarrow \boldsymbol{function\ of\ time\ difference\ only}.$$

$R_{XY}(t_1, t_2) = E[X(t_1)Y(t_2)]$ Represents the ***cross-correlation*** function of *X*(*t*) and *Y*(*t*) .

- We also define the ***auto-covariance*** function by:

$$C_{XX}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}]$$

- The ***cross-covariance*** function is defined by:

$$C_{XY}(t_1, t_2) = E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}]$$

3.6 N-order and strict-sense stationary:

- A random process is strict stationary to order *N* if the N-th order density function is invariant to time origin shift:

$$f_X(x_1, \ldots\ldots, x_N; t_1, \ldots\ldots, t_N) = f_X(x_1, \ldots\ldots, x_N; t_1 + \Delta, \ldots\ldots, t_N + \Delta)$$

for all $t_1, \ldots\ldots, t_N$ and Δ.

3.7 Time averages and ergodicity

- The ***time average*** of a quantity [.] is defined as:

$$A[.] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [.]dt$$

Where A is used to denote the time average as E for statistical averages.

- Here, we are interested in:
  - ✓ Mean value of a sample function:

$$\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

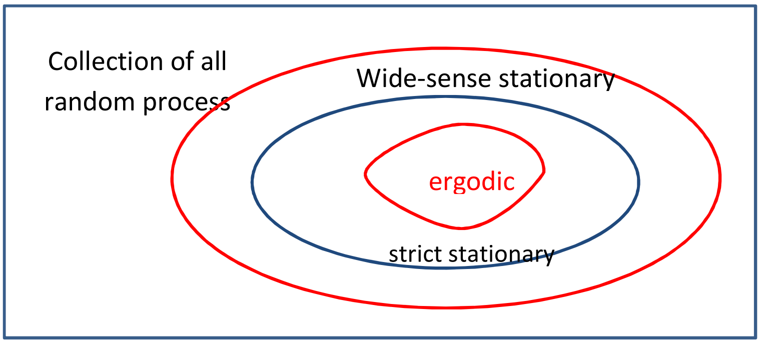
  - ✓ Time auto correlation function:

$$\mathcal{R}_{XX}(\tau) = A[x(t)x(t+\tau)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

- **Ergodic process:** A random process *X*(*t*) is said to be ergodic if the time averages $\bar{x}$ and $\mathcal{R}_{XX}(\tau)$ equal the statistical averages $\bar{X}$ and $R_{XX}(\tau)$. In other words, time averages equal the corresponding statistical averages.

  - ✓ **Erogodicity in the mean:** A random process that satisfies $\bar{x} = \bar{X}$, i.e, time average equals the ensemble average, is called mean-ergodic random process.

✓ **Erogodicity in the autocorrelation:** A random process that satisfies $\mathcal{R}_{XX}(\tau) = R_{XX}(\tau)$ is called autocorrelation-ergodic.



Example 3:

Let two random processes given by:

$$I(t) = \mathrm{X}\cos wt + Y\ \sin wt$$

$$Q(t) = \mathrm{X}\cos wt - \mathrm{Y}\cos wt$$

Where X and Y are two random variables with zero mean, uncorrelated, and each has a variance equal to σ^2 . Find the cross-correlation function between I(t) and Q(t)

Solution:

We have E[X] = E[Y] = 0

X and Y are uncorrelated ⇒E[XY] = 0

Variance= E[X^2] = E[Y^2] =σ^2

$R_{IQ}(\mathrm{t}_1,\mathrm{t}_2) = \mathrm{E}[\mathrm{I}(t_1)\ \mathrm{Q}(\mathrm{t}_2)]$

$= \ E[(X\cos wt_1 + Y\sin wt_1)(Y\cos wt_2 - X\cos wt_2)]$

0

$= E[XY](\cos wt_1 \cos wt_2 - \sin wt_1 \cos wt_2) + E[Y^2] \sin wt_1 \cos wt_2 - E[X^2] \cos wt_1 \cos wt_2$

$= \sigma^2 \cos wt_2 (\sin wt_1 - \cos wt_1)$

Let $\tau = t_2 - t_1 \Rightarrow t_2 = t_1 + \tau$

$R_{XY}(t, t+\tau) = \sigma^2 \cos w(t+\tau) [\sin wt - \cos wt]$

Example 4:

Let X(t) a random process with: $X(t) = A\cos(w_0 t + \theta)$

Where A and w_0 are constants and θ is random variable uniformly distributed in the interval [0,2π].

Is X(t) ergodic in the mean and autocorrelation function?

Solution:

We found that $E[X(t)] = 0 = \bar{X}$

$$R_{XX}(\tau) = \frac{A^2}{2}\cos(w_0\tau)$$

where τ is a constant

Now, $\bar{x} = A[x(t)] = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\ \cos(w_0 t + \theta)\, dt$

$$= \frac{A}{w_0} \lim_{T\to\infty} \frac{1}{2T} \sin(w_0 t + \theta)\Big|_{-T}^{T} = 0$$

$\bar{X} = \bar{x} \Rightarrow X(t)$ is ergodic in the mean.

$$\mathcal{R}_{XX}(t, t+\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} A\ \cos(w_0 t + \theta) . A\ \cos(w_0(t+\tau) + \theta)\, dt$$

$$= \frac{A^2}{2} \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} [\cos(w_0\tau) + \cos(2w_0 t + w_0\tau + 2\theta)]\, dt$$

$$= \frac{A^2}{2}\cos(w_0\tau)\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} dt = \frac{A^2}{2}\cos(w_0\tau) = R_{XX}(\tau)$$

$\mathcal{R}_{xx}(\tau) = R_{xx}(\tau) \Rightarrow X(t)$ is ergodic in the autocorrelation

4. Spectral characteristics

- In the previous chapter we introduced random processes and presented the related temporal characteristics.
- In this chapter we will present another aspect for representing random processes in the frequency domain.
- In this chapter, we assume that the random processes are wide sense stationary.

5. Power spectral density

5.1 Deterministic signals

- We known that the Fourier transform of a deterministic signal $s(t)$ is given by:

$$S(\omega) = \int_{-\infty}^{+\infty} s(t)e^{-j\omega t}dt$$

- $S(\omega)$ is called sometimes the spectrum of $s(t)$**.**
- In going for the time domain description $s(t)$ to the frequency domain $S(\omega)$ no information about the signal is lost.
- Which means that $S(\omega)$ forms a complete description of $s(t)$ and vise-versa.
- The signal $s(t)$ can be obtained using the inverse furrier transform:

$$s(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)e^{j\omega t}d\omega$$

Or

$$s(t) = \int_{-\infty}^{+\infty} S(f)e^{j2\pi ft}df$$

5.2 Random process

- If the random process $X(t)$ is stationary in the wide sense, then the power spectral density $S_{XX}(\omega)$ can be expressed as the Fourier transform of the autocorrelation function $R_{XX}(\tau)$:

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau$$

Or

$$S_{XX}(f) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j2\pi f\tau}d\tau$$

- As for deterministic signals, the autocorrelation function $R_{XX}(\tau)$can be obtained from the power spectral density $S_{XX}(f)$ using the inverse Fourier transform:

$$R_{XX}(\tau) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)e^{j\omega\tau}d\omega$$

Or

$$R_{XX}(\tau) = \int_{-\infty}^{+\infty} S_{XX}(f)e^{j2\pi f\tau}df$$

- The transformation $R_{XX}(\tau) \leftrightarrow S_{XX}(\omega)$ is sometimes called the ***wiener-Kinchin*** relations.
- The ***average power*** is given by:

$$P_{XX} = \int_{-\infty}^{+\infty} S_{XX}(f)df$$

$$P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega$$

- Note also the P_{XX} is given by:

$$P_{XX} = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{+\infty} E[X^2(t)]dt$$

5.3 Proprieties of the power density spectrum

- The power spectrum density has several proprieties:
  - ✓ $S_{XX}(\omega) \geq 0$
  - ✓ $S_{XX}(-\omega) = S_{XX}(\omega)$ for $X(t)$ real
  - ✓ $S_{XX}(\omega)$ is real
  - ✓ $P_{XX} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega)d\omega = \langle E[X^2(t)]\rangle$

Example 4

Let $X(t)$ be a wide sense stationary process with autocorrelation function

$$R_{XX}(\tau) = \begin{cases} A(1-\frac{|\tau|}{T}) & -T \leq \tau \leq T \\ 0 & otherwise \end{cases}$$

Where $T > 0$ and A are constants.

Determine the power spectrum density.

Solution:

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau)e^{-j\omega\tau}d\tau = \int_{-T}^{+T} A(1-\frac{|\tau|}{T})e^{-j\omega\tau}d\tau$$

We have

$$\int_{-T}^{+T} Ae^{-j\omega\tau}d\tau = -\frac{A}{j\omega}e^{-j\omega\tau}\Big|_{-T}^{T} = \frac{2A}{\omega}\sin(\omega T)$$

And $\int_{-T}^{+T} A\frac{|\tau|}{T}e^{-j\omega\tau}d\tau = \frac{A}{T}2\int_0^T \tau e^{-j\omega\tau}d\tau$

We have,

$$\int xe^{cx}dx = e^{cx}[\frac{x}{c} - \frac{1}{c^2}]$$

$$\frac{A}{T}2\int_0^T \tau e^{-j\omega\tau}d\tau = \frac{2A}{T\omega^2}[1-(1-j\omega T)e^{-j\omega T}]$$

$$S_{XX}(\omega) = \frac{2A}{\omega}\sin(\omega T) + \frac{2A}{T\omega^2}[1-(1-j\omega T)e^{-j\omega T}]$$

Fourier Transform of some common functions:

$x(t)$	$X(f)$	$X(\omega)$
$\delta(t)$	1	1
1	$\delta(f)$	$2\pi\delta(\omega)$
$\delta(t-t_0)$	$e^{-j2\pi f t_0}$	$e^{-j\omega t_0}$
$e^{j2\pi f_0 t}$	$\delta(f-f_0)$	$2\pi\delta(\omega-\omega_0)$
$\cos(2\pi f_0 t)$	$\frac{1}{2}\left[\delta(f-f_0)+\delta(f+f_0)\right]$	$\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$
$\sin(2\pi f_0 t)$	$\frac{1}{2j}\left[\delta(f-f_0)-\delta(f+f_0)\right]$	$-j\pi\left[\delta(\omega-\omega_0)-\delta(\omega+\omega_0)\right]$
$rect(t)$	$\mathrm{sinc}(f)$	$\mathrm{sinc}\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sinc}(t)$	$rect(f)$	$rect\left(\frac{\omega}{2\pi}\right)$
$\Lambda(t)$	$\mathrm{sinc}^2(f)$	$\mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$
$\mathrm{sinc}^2(t)$	$\Lambda(f)$	$\Lambda\left(\frac{\omega}{2\pi}\right)$
$e^{-\alpha t}u(t),\quad \alpha>0$	$\frac{1}{\alpha+j2\pi f}$	$\frac{1}{\alpha+j\omega}$
$te^{-\alpha t}u(t),\quad \alpha>0$	$\frac{1}{(\alpha+j2\pi f)^2}$	$\frac{1}{(\alpha+j\omega)^2}$
$e^{-\alpha\lvert t\rvert},\quad \alpha>0$	$\frac{2\alpha}{(\alpha^2+(2\pi f)^2}$	$\frac{2\alpha}{(\alpha^2+(\omega)^2}$
$e^{-\pi t^2}$	$e^{-\alpha f^2}$	$e^{-\alpha f^2}$
$\mathrm{sgn}(t)$	$\frac{1}{j\pi f}$	$\frac{2}{j\omega}$
$u(t)$	$\frac{1}{2}\delta(f)+\frac{1}{j2\pi f}$	$\pi\delta(\omega)+\frac{1}{j\omega}$
$\frac{d}{dt}\delta(t)$	$j2\pi f$	$j\omega$
$\sum_{n=-\infty}^{\infty}\delta(t-nT_0)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(f-\frac{n}{T_0}\right)$	$\frac{1}{T_0}\sum_{n=-\infty}^{\infty}\delta\left(\omega-\frac{2\pi n}{T_0}\right)$