Random process (or stochastic process)

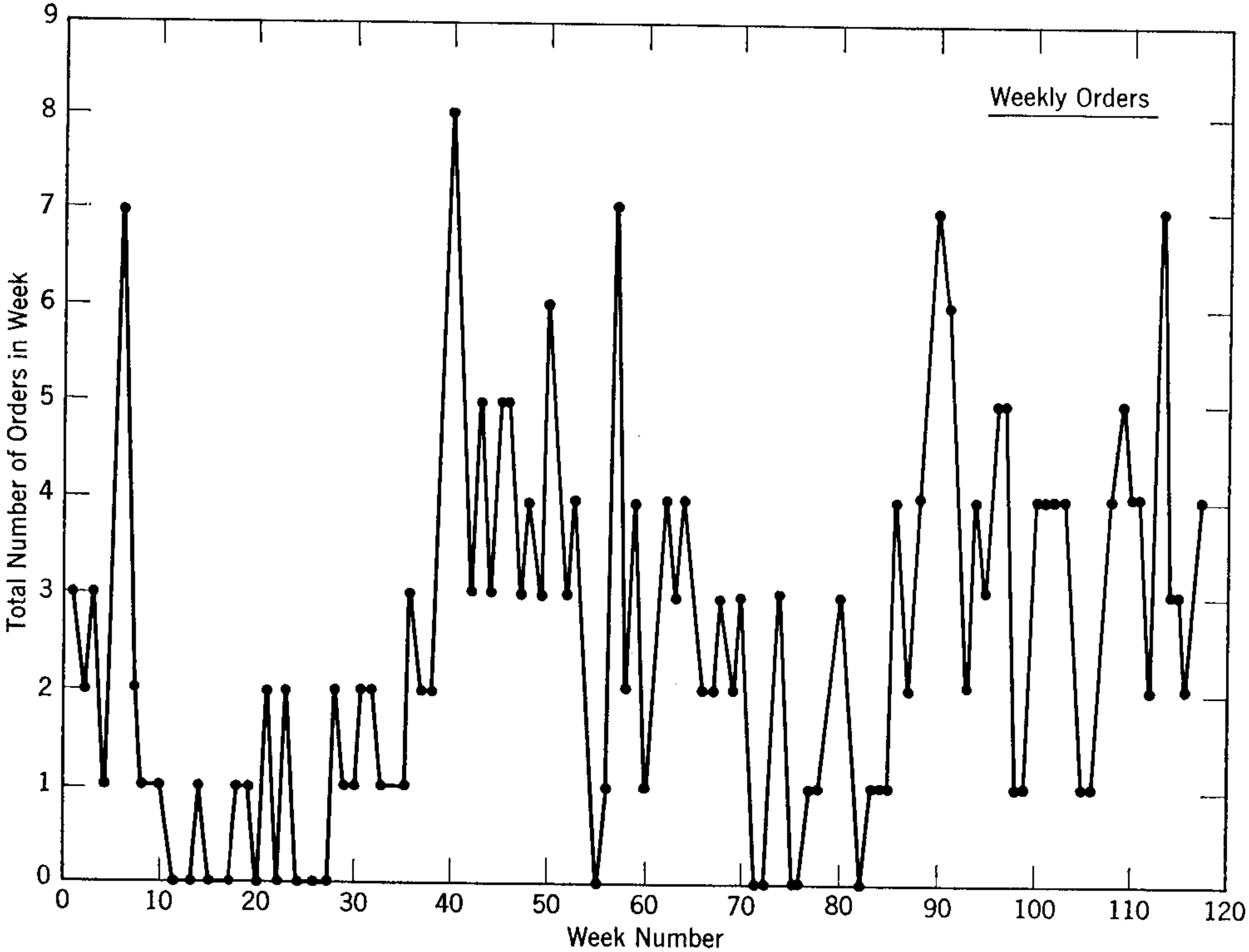
In many real life situation, observations are made over a period of time and they are influenced by random effects, not just at a single instant but throughout the entire interval of time or sequence of times.

In a "rough" sense, a random process is a phenomenon that varies to some degree unpredictably as time goes on. If we observed an entire time-sequence of the process on several different occasions, under presumably "identical" conditions, the resulting observation sequences, in general, would be different.

A random variable (RV) is a rule (or function) that assigns a real number to every outcome of a random experiment, while a random process is a rule (or function) that assigns a time function to every outcome of a random experiment.

Example

Observe the demand per week for certain service over time.



Example

The closing price of HSBC stock observed from June 15 to Dec. 22, 2003.

Example

Outcomes from Mark Six in Years 2000, 2001. We have two sequences of observations made twice per week. The actual sequence of observation is called the *realization* of the random process associated with the random experiment of Mark Six. The realizations in different years should differ, though the nature of the random experiment remains the same (assuming no change to the rule of Mark Six).

A random experiment may lead not only to a single random variable, but to an entire sequence

$$\{X_i : i = 1, 2, 3, \cdots\} = \{X_1, X_2, X_3, \cdots\}$$

of random variables (*indexed family* of random variables).

Consider the random experiment of tossing a dice at $t = 0$ and observing the number on the top face. The sample space of this experiment consists of the outcomes $\{1, 2, 3, \cdots, 6\}$. For each outcome of the experiment, let us arbitrarily assign a function of time $t\,(0 \leq t < \infty)$ in the following manner.

Outcome:	1	2	3	4	5	6
Function of time:	$x_1(t) = -4$	$x_2(t) = -2$	$x_3(t) = 2$	$x_4(t) = 4$	$x_5(t) = -t/2$	$x_6(t) = t/2$

The set of functions $\{x_1(t), x_2(t), \cdots, x_6(t)\}$ represents a random process.

Definition: A random process is a collection (or ensemble) of RVs $\{X(s, t)\}$ that are functions of a real variable, namely time t where $s \in S$ (sample space) and $t \in T$ (parameter set or index set).

The set of possible values of any individual member of the random process is called *state space*. Any individual member itself is called a *sample function* or a realisation of the process.

Classification of Random Processes

Depending on the continuous or discrete nature of the state space S and parameter set T, a random process can be classified into four types:

1. If both T and S are discrete, the random process is called a *discrete random sequence*. For example, if X_n represents the outcome of the nth toss of a fair dice, then $\{X_n, n \geq 1\}$ is a discrete random sequence, since $T = \{1, 2, 3, \cdots\}$ and $S = \{1, 2, 3, 4, 5, 6\}$.

2. If T is discrete and S is continuous, the random process is called a *continuous random sequence*.

   For example, if X_n represents the temperature at the end of the nth hour of a day, then $\{X_n, 1 \leq n \leq 24\}$ is a continuous random sequence, since temperature can take any value in an interval and hence continuous.

3. If T is continuous and S is discrete, the random process is called a *discrete random process*.

   For example, if $X(t)$ represents the number of telephone calls received in the interval $(0, t)$ then $\{X(t)\}$ is a discrete random process, since $S = \{0, 1, 2, 3, \cdots\}$.

4. If both T and S are continuous, the random process is called a *continuous random process*. For example, if $X(t)$ represents the maximum temperature at a place in the interval $(0, t), \{X(t)\}$ is a continuous random process. In the names given above, the word 'discrete' or 'continuous' is used to refer to the nature of S and the word 'sequence' or 'process' is used to refer to the nature of T.

Specifying a random process

Let $X_1, X_2, \cdots X_k$ be the k random variables obtained by sampling the random process $X(t, \zeta)$ at times $t_1, t_2, \cdots, t_k$:

$$X_1 = X(t_1, \zeta), X_2 = X(t_2, \zeta), \cdots, X_k = X(t_k, \zeta).$$

The joint behavior of the random process at these k time instants is specified by the joint cumulative distribution of the vector random variable $(X_1, X_2, \cdots, X_k)$.

A stochastic process is specified by the collection of kth-order joint cumulative distribution functions:

$$F_{X_1 \cdots X_k}(x_1, x_2, \cdots, x_k) = P[X_1 \leq x_1, X_2 \leq x_2, \cdots, X_k \leq x_k],$$

for any k and any choice at sampling instants $t_1, \cdots, t_k$.

If the stochastic process is discrete-valued, then a collection of probability mass functions can be used to specify the stochastic process:

$$P_{X_1, \cdots X_k}(x_1, x_2, \cdots, x_k) = P[X_1 = x_1, X_2 = x_2, \cdots, X_k = x_k].$$

Mean $m_X(t)$ of a random process $X(t)$ is

$$m_X(t) = E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x)\ dx.$$

In general, $m_X(t)$ is a function of time. Suppose we write

$$X(t) = m_X(t) + Y(t)$$

then $Y(t)$ has zero mean. Trends in the behavior of $X(t)$ are reflected in the variation of $m_X(t)$ with time.

Autocorrelation $R_X(t_1, t_2)$ of a random process $X(t)$ is the joint moment of $X(t_1)$ and $X(t_2)$

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{X(t_1),X(t_2)}\ dxdy.$$

Note that $f_{X_1(t),X_2(t)}$ is the second order pdf of $X(t)$ and $R_X(t_1, t_2)$ is a function of t_1 and t_2.

Autocovariance $C_X(t_1, t_2)$ of a random process $X(t)$ is defined as the covariance of $X(t_1)$ and $X(t_2)$:

$$\begin{aligned} C_X(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{X(t_2) - m_X(t_2)\}] \\ &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2). \end{aligned}$$

In particular, when $t_1 = t_2 = t$, we have

$$\mathsf{VAR}[X(t)] = E[(X(t) - m_X(t))^2] = C_X(t, t).$$

Correlation coefficient of $X(t)$ is defined as

$$\rho_X(t_1, t_2) = \frac{C_X(t_1, t_2)}{\sqrt{C_X(t_1, t_1)}\sqrt{C_X(t_2, t_2)}}; \quad |\rho_X(t_1, t_2)| \leq 1.$$

The mean, autocorrelation and autocovariance functions provide only partial description of a random process.

Example *Sinusoid with random amplitude*

Let $X(t) = A\cos 2\pi t, A$ is some random variable;

$$m_X(t) = E[A\cos 2\pi t] = E[A]\cos 2\pi t.$$

Remark The process is always zero for those values of t where $\cos 2\pi t = 0$.

$$\text{The autocorrelation is } R_X(t_1,t_2) = E[A\cos 2\pi t_1 A\cos 2\pi t_2] = E[A^2]\cos 2\pi t_1 \cos 2\pi t_2,$$

and

$$\begin{aligned}\text{autocovariance is } C_X(t_1,t_2) &= R_X(t_1,t_2) - m_X(t_1)m_X(t_2)\\ &= \{E[A^2] - E[A]^2\}\cos 2\pi t_1 \cos 2\pi t_2\\ &= \text{VAR}[A]\cos 2\pi t_1 \cos 2\pi t_2.\end{aligned}$$

Example *Sinusoid with random phase*

$$X(t) = \cos(\omega t + \Theta), \Theta \text{ is uniformly distributed in } (-\pi, \pi).$$

Mean

$$m_X(t) = E[\cos(\omega t + \Theta)] = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(\omega t + \theta)\ d\theta = 0.$$

The autocorrelation and autocovariance are

$$\begin{aligned} C_X(t_1, t_2) &= R_X(t_1, t_2) = E[\cos(\omega t_1 + \Theta)\cos(\omega t_2 + \Theta)] \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1}{2}\{\cos[\omega(t_1 - t_2)] + \cos[\omega(t_1 + t_2) + 2\theta]\}\ d\theta \\ &= \frac{1}{2}\cos(\omega(t_1 - t_2)). \end{aligned}$$

$m_X(t)$ is a constant and $C_X(t_1, t_2)$ depends only on $|t_1 - t_2|$.

Example *Discrete-time random process*

A discrete-time random process is defined by $X_n = s^n$, for $n \geq 0$, where s is selected at random from the interval $(0, 1)$.

(a) Find the cdf of X_n.

(b) Find the joint cdf for X_n and X_{n+1}.

(c) Find the mean and autovariance functions of X_n.

Solution

(a) For $0 < y < 1$, $P[X_n \leq y] = P[s^n \leq y] = P[s \leq y^{1/n}] = y^{1/n}$, since s is selected at random in $(0, 1)$.

(b) $P[X_n \le r, X_{n+1} \le t] = P[s^n \le r, s^{n+1} \le t] = P[s \le r^{1/n}, s \le t^{1/(n+1)}]$

$$= P[s \le \min(r^{1/n}, t^{1/(n+1)}] = \min(r^{1/n}, t^{1/(n+1)}).$$

(c)

$$\begin{aligned} E[X_n] &= \int_0^1 s^n \, ds \\ &= \left[\frac{s^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}. \\ R_X(n, n+k) &= E[X_n X_{n+k}] = \int_0^1 s^{2n+k} \, ds \\ &= \left[\frac{s^{2n+k+1}}{2n+k+1}\right]_0^1 = \frac{1}{2n+k+1}. \end{aligned}$$

$$\begin{aligned} C_X(n, n+k) &= R_X(n, n+k) - E[X_n]E[X_{n+k}] \\ &= \frac{1}{2n+k+1} - \left(\frac{1}{n+1}\right)\left(\frac{1}{n+k+1}\right). \end{aligned}$$

Example *Sum of independent random variables*

Let $Z(t) = At + B$, where A and B are *independent* random variables.

(a) Find the pdf of $Z(t), t > 0$.

(b) Find $m_Z(t)$ and $C_Z(t_1, t_2), t > 0$ and $t_1 > 0, t_2 > 0$.

Solution

(a) Since A and B are independent, so do At and B. Let $f_A(x)$ and $f_B(y)$ denote the pdf of A and B, respectively, then

$$f_{At}(x') = \frac{1}{t} f_A\left(\frac{x'}{t}\right).$$

$f_{Z(t)}(z)$ is given by the convolution between $f_{At}(x)$ and $f_B(x)$:

$$f_{Z(t)}(z) = \int_{-\infty}^{\infty} \frac{1}{t} f_A\left(\frac{x'}{t}\right) f_B(z - x')\ dx'.$$

(b) $m_Z(t) = E[At + B] = tE[A] + E[B]$.

$$\begin{aligned} C_Z(t_1, t_2) &= \text{COV}(t_1 A + B, t_2 A + B) \\ &= t_1 t_2 \text{COV}(A, A) + (t_1 + t_2)\text{COV}(A, B) + \text{COV}(B, B) \\ &= t_1 t_2 \text{VAR}(A) + \text{VAR}(B) + (t_1 + t_2)\text{COV}(A, B). \end{aligned}$$

Two random processes

The processes $X(t)$ and $Y(t)$ are said to be **independent** if the vector random variables $(X(t_1), \cdots, X(t_k))$ and $(Y(t'_1), \cdots, Y(t'_j))$ are independent for all k, j, and all choices of $t_1, \cdots, t_k$ and $t'_1, \cdots, t'_j$.

Cross-covariance

$$\begin{aligned} C_{X,Y}(t_1, t_2) &= E[\{X(t_1) - m_X(t_1)\}\{Y(t_2) - m_Y(t_2)\}] \\ &= R_{X,Y}(t_1, t_2) - m_X(t_1)m_Y(t_2) \end{aligned}$$

$X(t)$ and $Y(t)$ are uncorrelated if $C_{X,Y}(t_1, t_2) = 0$ for all t_1 and t_2.

Example *Sum of independent and identically distributed Gaussian random variables*

Let $X(t) = A\cos\omega t + B\sin\omega t$, where A and B are iid Gaussian random variables with zero mean and variance σ^2. Find the mean and autocovariance of $X(t)$.

$$E[X(t)] = m_X(t) = E[A]\cos\omega t + E[B]\sin\omega t = 0$$

since A and B both have zero mean.

$$\begin{aligned}
C_X(t_1, t_2) &= R_X(t_1, t_2) \\
&= E[(A\cos\omega t_1 + B\sin\omega t_1)(A\cos\omega t_2 + B\sin\omega t_2)] \\
&= E[A^2\cos\omega t_1\cos\omega t_2 + AB(\sin\omega t_1\cos\omega t_2 + \cos\omega t_1\sin\omega t_2) \\
&\quad + B^2\sin\omega t_1\sin\omega t_2] \\
&= \sigma^2[\cos\omega t_1\cos\omega t_2 + \sin\omega t_1\sin\omega t_2] \\
&= \sigma^2\cos\omega(t_1 - t_2).
\end{aligned}$$

Note that $E[A^2] = E[B^2] = \sigma^2$ and $E[AB] = E[A]E[B] = 0$.

Independent and identically distributed discrete time random processes

Let X_n be a discrete time random process consisting of a sequence of independent, identically distributed (iid) random variables with common cdf $F_X(x)$, mean m and variance σ^2. The joint cdf for any time instants $n_1, \cdots, n_k$ is given by

$$\begin{aligned} F_{X_1 \cdots X_k}(x_1, x_2, \cdots, x_k) &= P[X_1 \leq x_1, \cdots, X_k \leq x_k] \\ &= F_X(x_1) F_X(x_2) \cdots F_X(x_k). \end{aligned}$$

$$m_X(n) = E[X_n] = m, \text{ for all } n;$$

$$C_X(n_1, n_2) = E[(X_{n_1} - m)(X_{n_2} - m)] = E[X_{n_1} - m]E[X_{n_2} - m] = 0, \quad n_1 \neq n_2.$$

If $n_1 = n_2$, then $C_X(n_1, n_2) = \sigma^2$.

We write $C_X(n_1, n_2) = \sigma^2 \delta_{n_1, n_2}$, where $\delta_{n_1,n_2} = \begin{cases} 1 & \text{if } n_1 = n_2 \\ 0 & \text{otherwise} \end{cases}$.

Independent and stationary increments

1. Random changes of the form $X_{t+h} - X_t$, for fixed $h > 0$, are called *increments* of the process.

2. If each set of increments, corresponding to non-overlapping collection of time intervals, is mutually independent, then X_t is said to be a process with *independent increments*.

   That is, for any k and any choice of sampling instants $t_1 < t_2 \cdots < t_k$, the random variables

   $$X(t_2) - X(t_1), \cdots, X(t_k) - X(t_{k-1})$$

   are independent random variables.

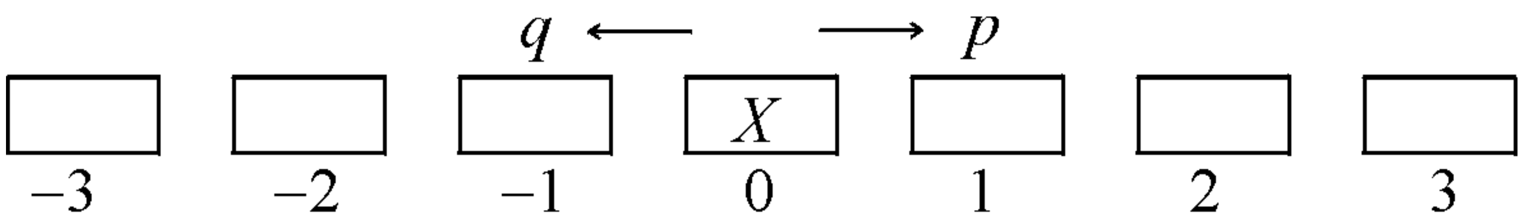
3. If $X_{t+h} - X_t$ has a distribution that depends only on h, not on t, then X_t is said to have *stationary increments*.

Markov process

A random process $X(t)$ is said to be Markov if the future of the process given the present is *independent of the past*. For discrete-valued Markov process

$$
\begin{aligned}
& P[X(t_k) = x_k | X(t_{k-1}) = x_{k-1}, \cdots, X(t_1) = x_1] \\
= \; & P[X(t_k) = x_k | X(t_{k-1}) = x_{k-1}].
\end{aligned}
$$

Example One dimensional random walk (frog hopping)



The initial position of a frog is taken to be at position 0. Let p and $q = 1 - p$ be the probabilities that the frog will choose to move to the right and to the left, respectively. Let X_n be the position of the frog after n moves.

$$P[X_1 = 1] = p, \quad P[X_1 = -1] = q$$

$$P[X_2 = 2] = p^2, \quad P[X_2 = 0] = 2pq, \quad P[X_2 = -2] = q^2$$

$$P[X_3 = 3] = P^3,\ P[X_3 = 1] = 3p^2q,\ P[X_3 = -1] = 3pq^2,\ P[X_3 = -3] = q^3$$

$\{X_1, \cdots, X_n\}$ is an indexed family of random variables. The random walk process is a discrete-valued discrete time random process.

Questions

1. Is this random process Markovian? Check whether only the information of the present position is relevant for predicting the future movements.

   Suppose the frog is at position 4 after 10 moves, does the probability that it will be in position 8 after 16 moves (6 more moves) depend on how it moves to position 4 within the first 10 moves?

   Answer The 11^{th} to 16^{th} moves are independent of the earlier 10 moves.

2. Are $X_{10} - X_4$ and $X_{16} - X_{12}$ independent? How about $X_{10} - X_4$ and $X_{12} - X_8$?

   Hint We are considering increments over non-overlapping and overlapping intervals, respectively.

Each move is an independent Bernoulli trial. The probability mass functions of X_3 are

$$P_{X_3}(3) = p^3, \quad P_{X_3}(1) = {}_3C_1 p^2 q, \quad P_{X_3}(-1) = {}_3C_2 p q^2, \quad P_{X_3}(-3) = q^3.$$

In general,

$$P_{X_k}(j) = {}_kC_{\frac{k+j}{2}} p^{\frac{k+j}{2}} q^{\frac{k-j}{2}}, \quad -k \leq j \leq k,$$

Why? Let R and L be the number of right moves and left moves, respecitvely. We have

$$R + L = k \quad \text{and} \quad R - L = j$$

so that $R = (k+j)/2$. Note that when k is odd (even), j must be odd (even).

How to find the joint pmf's?

For example, $P[X_2 = 0, X_3 = 3] = P[X_3 = 3 | X_2 = 0] P[X_2 = 0] = 0$;

$$P[X_2 = 2, X_3 = 1] = P[X_3 = 1 | X_2 = 2] P[X_2 = 2] = qp^2.$$

Sum processes

$S_n = \sum_{i=1}^{n} X_i, \quad X_i$'s are iid random variables. With $S_n = S_{n-1} + X_n, n = 2, 3, \cdots$ and $S_1 = X_1$, we take $S_0 = 0$ for notational convenience.

- It is called a binomial counting process if X_i's are iid Bernoulli random variables.

1. S_n is *Markovian*:

$$\begin{aligned} & P[S_n = \alpha_n | S_{n-1} = \alpha_{n-1}] \\ = \; & P[S_n = \alpha_n | S_{n-1} = \alpha_{n-1}, S_{n-2} = \alpha_{n-2}, \cdots, S_1 = \alpha_1]. \end{aligned}$$

   This is because $S_n = S_{n-1} + X_n$ and the value taken by X_n is independent of the values taken by $X_1, \cdots, X_{n-1}$.

2. S_n has *independent increments* in non-overlapping time intervals (no X_n's are in common).

3. S_n has *stationary increments*

   $P[S_{n+k} - S_k = \beta] = P[S_n - S_0 = \beta] = P[S_n = \beta]$, independent of k.

   This is because $S_{n+k} - S_k =$ sum of n iid random variables.

Remark

S_n and S_m are *not independent* since

$$S_n = X_1 + \cdots + X_n \quad \text{and} \quad S_m = X_1 + \cdots + X_m.$$

Say, for $n > m$, both contain $X_1, \cdots, X_m$.

Example Find $P[S_n = \alpha, S_m = \beta], \quad n > m.$

Solution

$$\begin{aligned} P[S_n = \alpha, S_m = \beta] &= P[S_n - S_m = \alpha - \beta, S_m = \beta] \\ &= P[S_n - S_m = \alpha - \beta] P[S_m = \beta], \end{aligned}$$

due to independent increments over non-overlapping intervals. Further, from stationary increments property, we have $S_n - S_m = S_{n-m} - S_0 = S_{n-m}$ so that

$$P[S_n = \alpha, S_m = \beta] = P[S_{n-m} = \alpha - \beta] P[S_m = \beta].$$

This verifies that S_n and S_m are not independent since

$$P[S_n = \alpha, S_m = \beta] \neq P[S_n = \alpha] P[S_m = \beta].$$

Mean, variance and autocovariance of sum processes

Let m and σ^2 denote the mean and variance of X_i, for any i. Since the sum process S_n is the sum of n iid random variables, its mean and variance are

$$\begin{aligned} m_S(n) &= E[S_n] = nE[X_i] = nm \\ \text{VAR}[S_n] &= n\text{VAR}[X_i] = n\sigma^2. \end{aligned}$$

Note that both $m_S(n)$ and $\text{VAR}[S_n]$ grow linearly with n. The autocovariance of S_n is

$$\begin{aligned} C_S(n,k) &= E[(S_n - E[S_n])(S_k - E[S_k])] \\ &= E\left[\left\{\sum_{i=1}^{n}(X_i - m)\right\}\left\{\sum_{j=1}^{k}(X_j - m)\right\}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{k} E[(X_i - m)(X_j - m)] \\ &= \sum_{i=1}^{\min(n,k)} C_X(i,i) = \min(n,k)\sigma^2 \end{aligned}$$

since $E[(X_i - m)(X_j - m)] = \sigma^2\delta_{i,j}$ and only those terms with $i = j$ survive.

Alternative method

Without loss of generality, we let $n \le k$ so that $n = \min(n, k)$.

$$\begin{aligned} C_S(n,k) &= E[(S_n - nm)(S_k - km)] \\ &= E[(S_n - nm)\{S_n - nm + (S_k - km) - (S_n - nm)\}] \\ &= E[(S_n - nm)^2] + E[(S_n - nm)(S_k - S_n - (k-n)m)], \end{aligned}$$

and since S_n and $S_k - S_n$ are independent, so

$$\begin{aligned} C_S(n,k) &= E[(S_n - nm)^2] + E[S_n - nm]E[S_k - S_n - (k-n)m] \\ &= E[(S_n - nm)^2] = \mathrm{Var}[S_n] = n\sigma^2 = \min(n,k)\sigma^2. \end{aligned}$$

Binomial counting process

Let S_n be the sum of n independent Bernoulli random variables, that is, S_n is a binomial random variable with parameters n and p. When there are j successes out of n trials, then $S_n = j$.

$$P[S_n = j] = \begin{cases} {}_nC_j p^j (1-p)^{n-j} & \text{for} \quad 0 \leq j \leq n \\ 0 & \text{otherwise} \end{cases} .$$

Note that S_n has mean np and variance $np(1-p)$.

To prove the claim on the variance of S_n, we consider

$\mathrm{VAR}[S_n] = E[S_n^2] - E[S_n]^2$, where $E[S_n^2] = \sum_{j=0}^{n} j^2{}_nC_j p^j q^{n-j}$.

Note that $\sum_{j=0}^{n} j^2{}_nC_j p^j q^{n-j} = \sum_{j=2}^{n} j(j-1)_nC_j p^j q^{n-j} + \sum_{j=0}^{n} j_nC_j p^j q^{n-j}$

$$= n(n-1)p^2 \sum_{j'=0}^{n-2} {}_{n-2}C_{j'} p^{j'} q^{n-2-j'} + np, \text{ where } j' = j-2.$$

Lastly, $\mathrm{VAR}[S_n] = n(n-1)p^2 + np - n^2p^2 = np - np^2 = npq$.

One-dimensional random walk revisited

mean $= E[X_n] = 0$

variance $= \text{Var}(X_n) = 4p(1-p)$

Hence, the autocovariance

$$C_X(n,k) = \min(n,k)4p(1-p).$$

Example Let X_n consist of an iid sequence of Poisson random variables with mean α.

(a) Find the pmf of the sum process S_n.

(b) Find the joint pmf of S_n and S_{n+k}.

(c) Find COV(S_{n+k}, S_n).

Solution

(a) Recall that the sum of independent Poisson variables remains to be Poisson distributed. Let α be the common mean of these independent Poisson variables. The sum of these n variables is a Poisson variable with mean

$$\mu = E[X_1] + \cdots + E[X_n] = n\alpha.$$

Now, S_n is a Poisson variable with mean $n\alpha$.

$$P[S_n = m] = \frac{(n\alpha)^m e^{-n\alpha}}{m!}.$$

(b) $P[S_n = m, S_{n+k} = \ell] = P[S_{n+k} = \ell | S_n = m] P[S_n = m], \quad \ell \geq m$

$$= P[S_k = \ell - m] P[S_n = m] \quad \text{(stationary increments)}$$

$$= \frac{(k\alpha)^{\ell - m} e^{-k\alpha}}{(\ell - m)!} \frac{(n\alpha)^m e^{-n\alpha}}{m!}.$$

(c) $\text{VAR}(S_n) = \text{VAR}(S_{n+k} - S_k) = \text{VAR}(S_{n+k}) + \text{VAR}(S_k) - 2\text{COV}(S_k, S_{n+k})$

so that $\text{COV}(S_k, S_{n+k}) = \frac{1}{2}[\text{VAR}(S_{n+k}) + \text{VAR}(S_k) - \text{VAR}(S_n)]$

$$= \frac{1}{2}[(n+k)\alpha + k\alpha - n\alpha] = k\alpha.$$